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Dynamics of partial control

Juan Sabuco,[1] Miguel A. F. Sanjuán,[1] and James A. Yorke[2]
[1]Departamento de Física, Universidad Rey Juan Carlos, Tulipán s/n, 28933 Móstoles, Madrid, Spain
[2]University of Maryland, College Park, Maryland 20742, USA

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Safe sets are a basic ingredient in the strategy of partial control of chaotic systems. Recently we have found an algorithm, the sculpting algorithm, which allows us to construct them, when they exist. Here we define another type of set, an asymptotic safe set, to which trajectories are attracted asymptotically when the partial control strategy is applied. We apply all these ideas to a specific example of a Duffing oscillator showing the geometry of these sets in phase space. The software for creating all the figures appearing in this paper is available as supplementary material. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4754874]

In our chaotic lives we usually do not try to specify our plans in great detail, or if we do, we should be prepared to make major modifications. Our plans for what we want to achieve are accompanied with situations we must avoid. Disturbances often disrupt our immediate plans, so we adapt to new situations. We only have partial control over our futures. Partial control aims at providing toy examples of chaotic situations where we try to avoid disasters, constantly revising our trajectories. More mathematically, partial control of chaotic systems is a new kind of control of chaotic dynamical systems in the presence of disturbances. The goal of "partial control" is to avoid certain undesired behaviors without determining a specific trajectory. The surprising advantage of this control technique is that it sometimes allows the avoidance of the undesired behaviors even if the control applied is smaller than the external disturbances of the dynamical system. A key ingredient of this technique is what we call *safe sets*. Recently we have found a general algorithm for finding these sets in an arbitrary dynamical system, if they exist. The appearance of these safe sets can be rather complex although they do not appear to have fractal boundaries. In order to understand better the dynamics on these sets, we introduce in this paper a new concept, the *asymptotic safe set*. Trajectories in the safe set tend asymptotically to the asymptotic safe set. We present two algorithms for finding such sets. We illustrate all these concepts for a time-2π map of the Duffing oscillator.

I. INTRODUCTION

Transient chaos[1,2] is a physical phenomenon which occurs in systems where trajectories behave chaotically for a finite amount of time in a compact (i.e., closed and bounded) region Q, until they move toward a final state. We study discrete-time dynamics where the final state is typically either a periodic state or the divergence of the system towards ∞. The topological structure inside Q that causes this behavior is a zero-measure set known as a *chaotic saddle*.[2,3] This kind of system is usually modeled by a continuous map f of phase space to itself

$$q_{n+1} = f(q_n), \tag{1}$$

where q_n is the state at time n and f the function that relates them. This map can also be the stroboscopic map or the Poincaré return map of some continuous system. The dimension of the phase space is arbitrary, but realistically our tools work quickly only in dimensions 1 and 2.

Manifestations of transient chaos are present in a wide variety of systems, in many of which it may be necessary to keep the orbits away from certain regions, that is, to keep trajectories from leaving the compact region Q in which we have transient chaotic behavior. Examples can be found in Refs. 4–6. The problem gets even more complicated when we consider the presence of disturbances in our systems and feedback controls.

Admissible trajectories. Thus, if we add the disturbances ξ_n followed by a feedback control u_n, our model becomes

$$q_{n+1} = f(q_n) + \xi_n + u_n \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{2}$$

We assume throughout this paper that

$$\xi_0 > u_0 > 0 \quad |\xi_n| \le \xi_0 \quad |u_n| \le u_0. \tag{3}$$

We call such ξ_n and u_n ***admissible***. Furthermore we will call a trajectory q_n that satisfies Eq. (2) an **admissible trajectory** when all ξ_n and u_n are admissible. In this equation, we assume that feedback control u_n is chosen with knowledge of q_n, ξ_n and the function f. Notice u_n cannot in general cancel out ξ_n because the control bound is smaller than the disturbance bound. In standard controlling-chaos situations, we would have the reverse.

We can also think of Eq. (2) as a two-person game where the first person chooses each ξ_n (knowing q_n and f) with the goal of forcing the admissible trajectory to leave Q. The second person chooses u_n with the goal of staying in Q. For the second player, this is a game of survival[7] for it is impossible to win at any finite n.

Safe sets. Now suppose there is a set Q for which our goal is to have the entire admissible trajectory q_n remain in Q (i.e., for all $n \ge 1$). We say a point $q_1 \in Q$ is **safe** if it has the property that no matter how the admissible disturbances are chosen, and there exists an admissible feedback control such that the entire (admissible) trajectory q_n of Eq. (2) remains in Q. Notice that q_2 is also a safe point since it is clear that a feedback control can be chosen so that it stays in Q. Similarly all the points q_n are safe (for $n \geq 1$). If a set $S \subset Q$ has the property that it is always possible to keep all $q_n \in S$ when $q_1 \in S$, we say S is **safe**. Let $S \subset Q$ be the set of all safe points. Then S is an example of a safe set. It is the largest safe set in Q. Notice that whether a trajectory or a set is safe depends on the choice of ξ_0 and u_0 as well as Q. Figure 1 shows a safe set along with a set which will be introduced later called "asymptotic safe set."

If we know a safe set S, then we can create a feedback control with the following strategy; namely, given $f(q_n) + \xi_n$, choose $u = u_n$ such that $f(q_n) + \xi_n + u \in S$. It follows that for any $q \in S$ and any admissible ξ, there is an admissible u such that

$$f(q) + \xi + u \in S. \tag{4}$$

We will refer to any admissible trajectory that stays in a safe set as a **safe trajectory**.

The following example is quite counterintuitive. Yet it is also the simplest example of partial control, so it is important to understand it.

An example of a safe set for the slope-three tent map. Let $f(x) := 3 - 3|x|$. The trajectory of almost every initial condition diverges to $-\infty$ for the process $x_{n+1} = f(x_n)$. That map f has an invariant Cantor set, namely, the middle-third Cantor set on $Q = [-3/2, +3/2]$. Let $u_0 = 1$ and $\xi_0 = 2$. Since almost every trajectory leaves Q and diverges to $-\infty$, it seems the control would be unable to keep the trajectory bounded. None the less, there is a safe set $S := \{-1, +1\}$; this is $f^{-1}(0)$, the set of points that map to the critical point $x = 0$. To see that this is safe, choose $q \in S$ and an admissible ξ. Then $f(q) = 0$ so $f(q) + \xi \in [-2, 2]$. Each point of that interval is within $u_0 = 1$ of S. Hence an admissible u can be chosen so that $f(q) + \xi + u \in S$. Note that S is not in the invariant Cantor set. When $\xi = 2$ and $u_0 < 1$, there is no safe set.

Notice that which point of S the trajectory goes to is determined by whether ξ is positive or negative. The control does not determine the trajectory. That is why this is partial control. In control theory, the control is used to implement a trajectory that is specified in advance.

Another example is the same map but with $u_0 = 1/3$ and $\xi_0 = 2/3$; there is a safe set $S := \pm 1 \pm 1/3$ which is $f^{-2}(0)$. Then $f(S) = f^{-1}(0) = \{\pm 1\}$.

All this is pointless if we cannot find S for cases more complicated than the one above. However, we have recently reported an algorithm[8] for finding the largest safe set in Q. This approach sometimes allows us to keep trajectories inside a region Q even when the maximum amplitude of the control u_0 is smaller than the maximum amplitude ξ_0 of the disturbances, even when almost every trajectory of the deterministic system (Eq. (1)) leaves Q. This is rather counterintuitive. We call this situation **partial control of chaos**,[9–11] and our method for finding the largest safe set in Q is what we call a sculpting algorithm; see Ref. 11. We called it a "sculpting" algorithm because we start with a larger set and successive carve away and discard parts of the set, asymptotically approaching the safe set. In this paper we are reporting on the long-term behavior of safe trajectories, and it turns out that to determine this asymptotic set, we use a different sculpting algorithm. To avoid complete confusion, we present our original sculpting algorithm (for sculpting safe sets), so we can compare it with the new sculpting algorithm for "asymptotic" sets, which we introduce in the next section. The codes to compute the safe sets and the asymptotic safe sets are available (see Ref. 12).

Numerical calculations. All results reported in this paper were made using a grid of 6000 by 6000 points in the square shown in Fig. 1. The map f is approximated by a map f^* where for a grid point p, we define $f^*(p)$ to be the nearest grid point to $f(p)$. If there is more than one closest grid point, then the one with the smallest coordinates is picked. The finer the grid, the more accurate the calculation, and obviously it takes longer time. We only report results that persist without significant change when we significantly increase the grid density.

We look for the minimum of the u_0 for which there is a safe set, which we denote as u_0^{min}. We are currently unable to determine the uncertainty in the smallest u_0^{min}. There is of course a huge change in the safe set as u_0 is decreased past u_0^{min}. When it is below u_0^{min}, no safe set exists. We do not know what critical event in the dynamics occurs at u_0^{min}.

We observe the safe set suddenly disappearing as u_0 is decreased. See Figs. 2 and 3. In practice it is important to

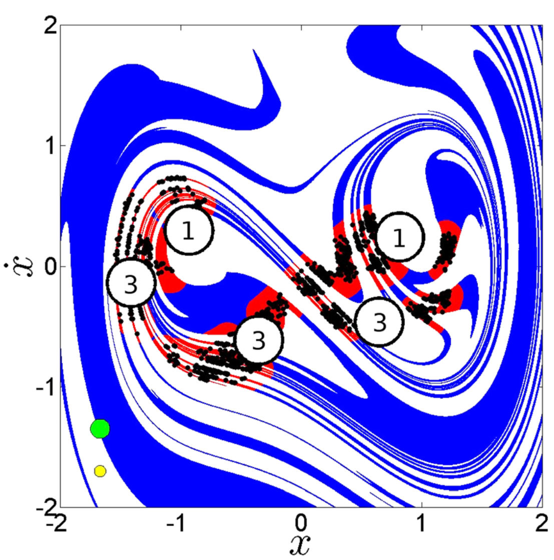

FIG. 1. Duffing safe set and an asymptotic (safe) set. All the figures in this paper are for where f is the time-2π map of the partially controlled Duffing oscillator $\ddot{x} + 015\dot{x} - x + x^3 = 0245 \sin(t)$ and $Q = [-2, 2] \times [-2, 2]$ minus five disks of radius 0.2 that are centered at the five points of the three attractors (1 denotes a fixed point and 3 a period-3 point). Furthermore $\xi_0 = 0.08$ and $u_0 = 0.0475$, except when u_0 takes multiple values. In each figure the green disk in the lower left has radius ξ_0, and the smaller yellow disk has radius u_0, shown to illustrate the scale. We have used a 6000 by 6000 point grid in the square shown. When all disturbances and control are 0, almost every trajectory appears to asymptote to one of the attracting periodic orbits. A sample admissible trajectory (that stays in the safe set) of 1000 iterates is shown as black dots. The red set (on which the black dots lie) is an "asymptotic safe set," which is described later in the paper. It includes the long-term parts of the trajectory.

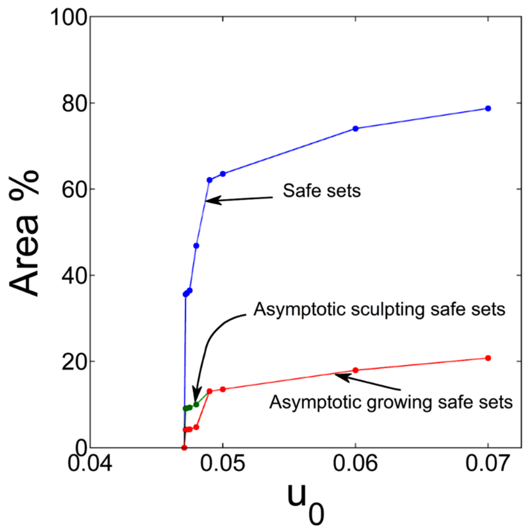

FIG. 2. Areas of sets depending on u_0. See Fig. 1 for the map f and parameters. Each curve represents the percentage of the area of the square $[-2,2]\times[-2,2]$ that it occupies. As u_0 varies, the highest curve is for the largest safe set S in Q. The lower two are for the asymptotic safe sets. Note that they are equal except on a small part of the range. The curves seem to be discontinuous at $u_0 = u_0^{min}$, $0.0471 < u_0^{min} < 0.0472$, below which no region is safe, and it is impossible to partially control the system. At $u_0 = 0.0472$ the upper curve is 35.6%, while the lower two curves are 4.1%. Recall that all calculated numbers will vary slightly with grid size and implementation of the algorithms, but we expect the patterns to persist.

keep $u_0 > u_0^{min}$ to avoid this numerical discontinuity. Numerical experiments to find u_0^{min} will result in slightly different values, depending on the grid size used and on the details of the implementation of the algorithm.

Duffing oscillator safe set. Throughout this paper we investigate the Duffing equation

$$\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\ \sin(t). \tag{5}$$

Let $q = (x, \dot{x})$ and let $f(q)$ be the time-2π map of the Duffing oscillator. With these parameters, f has three attractors (two fixed points and one period 3 orbit). The approximate positions of the fixed points are (0.815, 0.242) and (–0.933, 0.299), while the approximate positions of the period 3 are $(-1.412,\ -0.137)$, $(-0.354,\ -0.614)$, and $(0.645,\ -0.464)$.[13] The parameters have been chosen so that their basins have the Wada property; that is, every point on the boundary of any of the three basins is also on the boundary of the other two basins. With these parameters, the Duffing oscillator has transient chaos due to a chaotic saddle on the boundary of the basins.

When we apply the partial control technique to the Duffing oscillator, our goal is to keep the trajectories away from the attracting fixed points and the attracting periodic orbit of period 3. The time-2π map of Eq. (5) exhibits transient chaotic behavior. The orbits behave chaotically, but it appears that almost every orbit falls close enough to one of the stable periodic attractors to be captured after some time.

With $\xi_0 = 0.08$, we have found that there is a safe set as long as $u_0 \geq 0.0472$, that is, with a ratio of $u_0/\xi_0 \approx 0.59$. When $u_0 = 0.0471$ there is no safe set.

II. ASYMPTOTIC SAFE SETS

Recall Q denotes the set that we want admissible trajectories to stay in. Let S denote the largest safe set in Q. We say a point p is a **predecessor** (or pre-image) of a point q and q is a **successor** of p if there exist admissible ξ and u for which

$$f(p) + \xi + u = q. \tag{6}$$

We say q is a ***S*-successor** if it is a successor of a point in S and it is in S. We say a non-empty set $A \subset S$ is an **asymptotic set for *Q*** if each $p \in A$ has a predecessor in A and each of p's S-successors is in A. Every safe trajectory that is in A for some time n must stay in A thereafter.

The red set in Fig. 1 denotes an asymptotic safe set in S that we generated using what we will later call a "growing" algorithm. The particular sample admissible trajectory (black dots) lies in A. No admissible trajectory that starts in an asymptotic set A can leave it unless it also leaves S.

We now give a sculpting algorithm for obtaining the largest asymptotic safe set S_∞ in S. Imagine all possible admissible trajectories q_n lying entirely in $S, (n > 0)$. For each $n > 0$, write S_n for the set of all possible values that q_n might have. Then $S_1 = S$ and $S_{n+1} \subset S_n$. Note that S_{n+1} is the set of S-successors of points in S_n. Write $S_\infty = \cap S_n$. Then S_∞ is asymptotic set and it is the largest in S. The compact sets S_n converge to S_∞. Therefore, for any admissible trajectory in S, the distance of q_n to S_∞ goes to 0 as $n \to \infty$. It would most likely eventually lie inside the asymptotic set.

How do we compute the sets S_n? For a compact set $C \subset S$, let

$$\rho_S(C) := \{q \text{ that are } S\text{-successors of points in } C\} \tag{7}$$

Typically S will have some points that have no predecessors in S. We now use ρ_S to characterize the above defined sets S_n. We have

$$S_1 := S; \quad S_{n+1} = \rho_S(S_n); \quad S_\infty = \cap_n S_n. \tag{8}$$

Notice that since S is safe, each point has a successor in S, so none of these sets is empty. Every S_n is safe. Since all S_n are compact and the sequence is nested, the intersection S_∞ is non empty. It follows that $S_\infty = \rho_S(S_\infty)$. In fact any set A that satisfies

$$A = \rho_S(A) \tag{9}$$

is an asymptotic set since all successors of each point in A are in A and each point in A has a predecessor in A. Furthermore each such set is safe, because each point $p \in A$ is also in S, so every S-successor is in A.

For any admissible trajectory that lies entirely in S, it follows that the distance $\mathrm{dist}(q_n, S_\infty) \to 0$, and we may find $q_n \in S_\infty$ for n sufficiently large. If $q_n \in S_\infty$ for some n, it must stay in S_∞ thereafter.

Procedure of Eq. (8) is a "sculpting algorithm" since we start with a large set S and successively cut parts away from S, each S_n smaller than S_{n-1}.

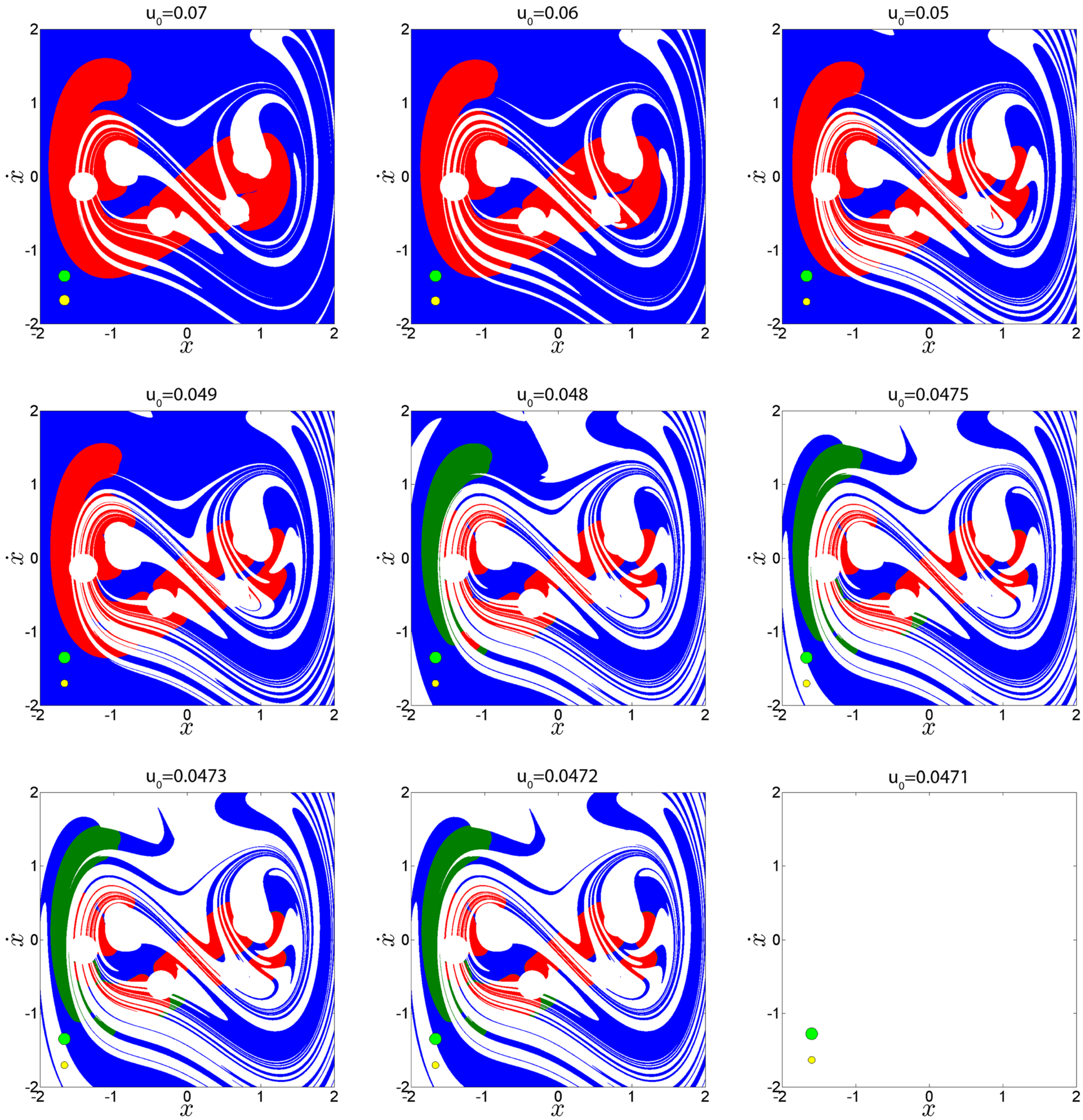

FIG. 3. The safe and asymptotic safe sets as u_0 decreases. The safe set is the non white region, blue + red + green. This figure shows in red the asymptotic growing safe set. The growing asymptotic set is sometimes smaller than the sculpting asymptotic set and sometimes equal. When they are not equal, the part of the sculpting set that is not in the growing asymptotic set is shown in dark green. At $u_0 = 0.0471$ no asymptotic safe set exists.

III. IMPLEMENTING THE SAFE-SET-SCULPTING ALGORITHM

For a set C, we define two set operations. First, for a specified value $u_0 \geq 0$, we define $C + u_0$ = "C fattened up by u_0" = $\{q$ whose distance from C is $\leq u_0\}$, as shown in Fig. 4(a). Likewise, for a value $\xi_0 \geq u_0$, we also define $(C + u_0) - \xi_0$ = "$(C + u_0)$ shrunken by ξ_0" = $\{q$ that are further than ξ_0 from the exterior of $(C + u_0)\}$, as in Fig. 4(b). We write $C + u_0 - \xi_0$ for $(C + u_0) - \xi_0$ which in general is not equal to $C + (u_0 - \xi_0)$.

In the tent-map example, $S + u_0 = [-2, +2]$. Then $S + u_0 - \xi_0 = \{0\}$. But $S + (u_0 - \xi_0) = \{\pm 1\} - 1$, which is the empty set. Hence $f(S) = \{0\} = S + u_0 - \xi_0$, so S is safe. It is also an asymptotic set, since safe trajectories can take on both values in S infinitely often.

Before describing an efficient algorithm for finding an asymptotic safe set within a safe set, we remind the reader of what we call the "safe-set sculpting algorithm" because of their similarity and their differences. See Ref. 8 where it is called the "sculpting algorithm."

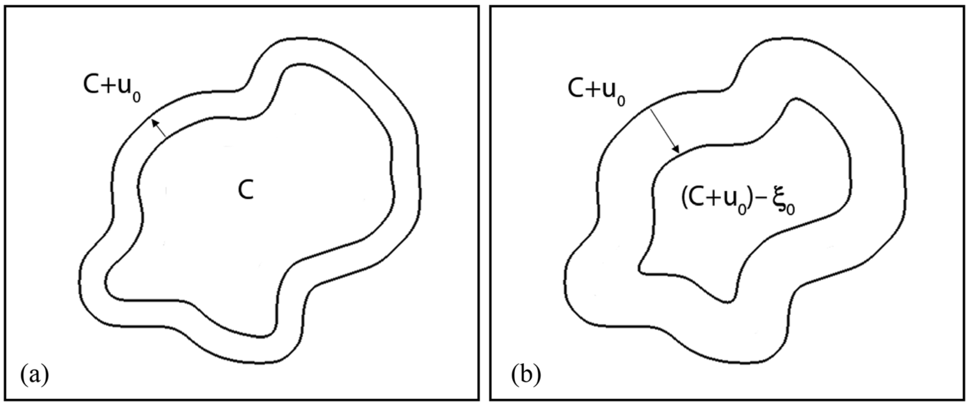

FIG. 4. Expanding and shrinking operators on sets. (a) First we "fatten" C by a distance u_0 obtaining $C+u_0$. (b) We "shrink" $C+u_0$ by ξ_0 to obtain $C+u_0-\xi_0$.

When we start sculpting a compact set C to find the largest safe set in C, with specified values u_0 for control and ξ_0 for disturbances, we consider that there will be good points and bad points. We say that a point p in C is "bad" (for C) if there exists an admissible ξ such that the distance of $f(p)+\xi$ from C is more than u_0. Such a point cannot be in a safe set in C. In that case $f(p)+\xi$ has no admissible control u for which $f(p)+\xi+u$ is in C. We call the rest of the points in C "good" (for C). We define the **safe-set-sculpting operator**

$$\Psi(C) := \{\text{good points} \in C\}. \tag{10}$$

It cuts away and discards the bad part of C. If C is closed, so is $\Psi(C)$. Points may be good for C but bad for $\Psi(C)$ and so have to be discarded in this iterative process. The sculpting algorithm says that given a compact set Q, we can generate its largest safe set Q_∞ as follows:

$$Q_1 := Q; \quad Q_{n+1} := \Psi(Q_n) \subset Q_n; \quad Q_\infty := \cap_{n=1}^{\infty} Q_n. \tag{11}$$

For compact Q, all Q_n and Q_∞ are compact. We leave it to the reader to show that Q_∞ **is a safe set**, the largest safe set in Q. Our first numerical implementation of this procedure required days for simple computations. Our current implementation is 30 000 times faster. We find the safe set by applying our safe-set sculpting algorithm to Q (the square minus the five balls in Fig. 1); we obtain the safe set in 15 iterations of Ψ for the specific choice of ξ_0 and u_0. The process stops because we use a finite grid. A perfect calculation would take an infinite number of iterates but would converge to a set which looks very much like the displayed safe set.

Implementing the sculpting algorithm. An equivalent formulation of Ψ is

$$\Psi(C) = \{p \in C \text{ such that } f(p) \in (C + u_0 - \xi_0)\}. \tag{12}$$

Notice that this formulation does not require testing each point with every admissible ξ, so it is much easier to implement (when restricting calculations to a grid). The safe-set sculpting algorithm applied to the time-2π map of the Duffing oscillator is shown in Fig. 5, although only 12 iterations out of the 15 appear. As it was mentioned earlier, there is a value of the control parameter u_0^{min} which corresponds to the smallest u_0 for which there is a safe set.

A. Sculpting the asymptotic safe set in a given safe set

It is not immediately clear from the definition of ρ_S how Eq. (7) can be implemented, since it appears we must experiment with all admissible ξ for each p in the set. Therefore we give an equivalent formulation. For a compact set $C \subset S$ where S is safe, the set of successors is $C + \xi_0 + u_0$. Hence the set of successors of points in C that lie in S is

$$\rho_S(C) = (f(C) + \xi_0 + u_0) \cap S.$$

Here it is clear how to (approximately) compute $\rho_S(C)$ when calculations are restricted to a grid.

Computing an asymptotic set. By repeating the application of $Q_{n+1} := \Psi(Q_n)$ (Eq. (11)), the algorithm "converges" in a finite number of steps (since we use a grid) to the asymptotic safe set. That is, we come to an n for which $Q_n = Q_{n+1}$. This is what we have computed with the Duffing oscillator. See Fig. 6 which shows the sculpting of the asymptotic safe set. We start with the safe set S found (using a grid) for the Duffing oscillator in Fig. 1. Then we sculpt that safe set using Eq. (8) until reaching its asymptotic safe set. We stop the algorithm at the step when no points are removed.

Asymptotic safe sets seem less dependent on the exact choice of Q as long as the set is in Q, although the asymptotic safe set could get bigger in some cases when Q is chosen larger.

B. Growing an asymptotic set

An asymptotic set A for a safe set S has the property

$$\rho_S(A) = A.$$

Suppose $p \in S$ satisfies $f(p) = p$. Write $B_1 := \{p\}$. Then since p is a fixed point, $B_2 := \rho_S(B_1) = (B_1 + u_0 + \xi_0) \cap S$. A key property here is that $B_1 \subset B_2$; later, we will weaken even that property. The **growing algorithm** is as follows:

$$\begin{aligned} &\text{Assume } B_1 \subset B_2; \quad \text{define } B_{n+1} := \rho_S(B_n); \\ &B_\infty := \text{ closure of } \cup_n B_n. \end{aligned} \tag{13}$$

Then B_∞ is asymptotic. The growing of an asymptotic safe set is shown in Fig. 7. It is also a safe set since each of

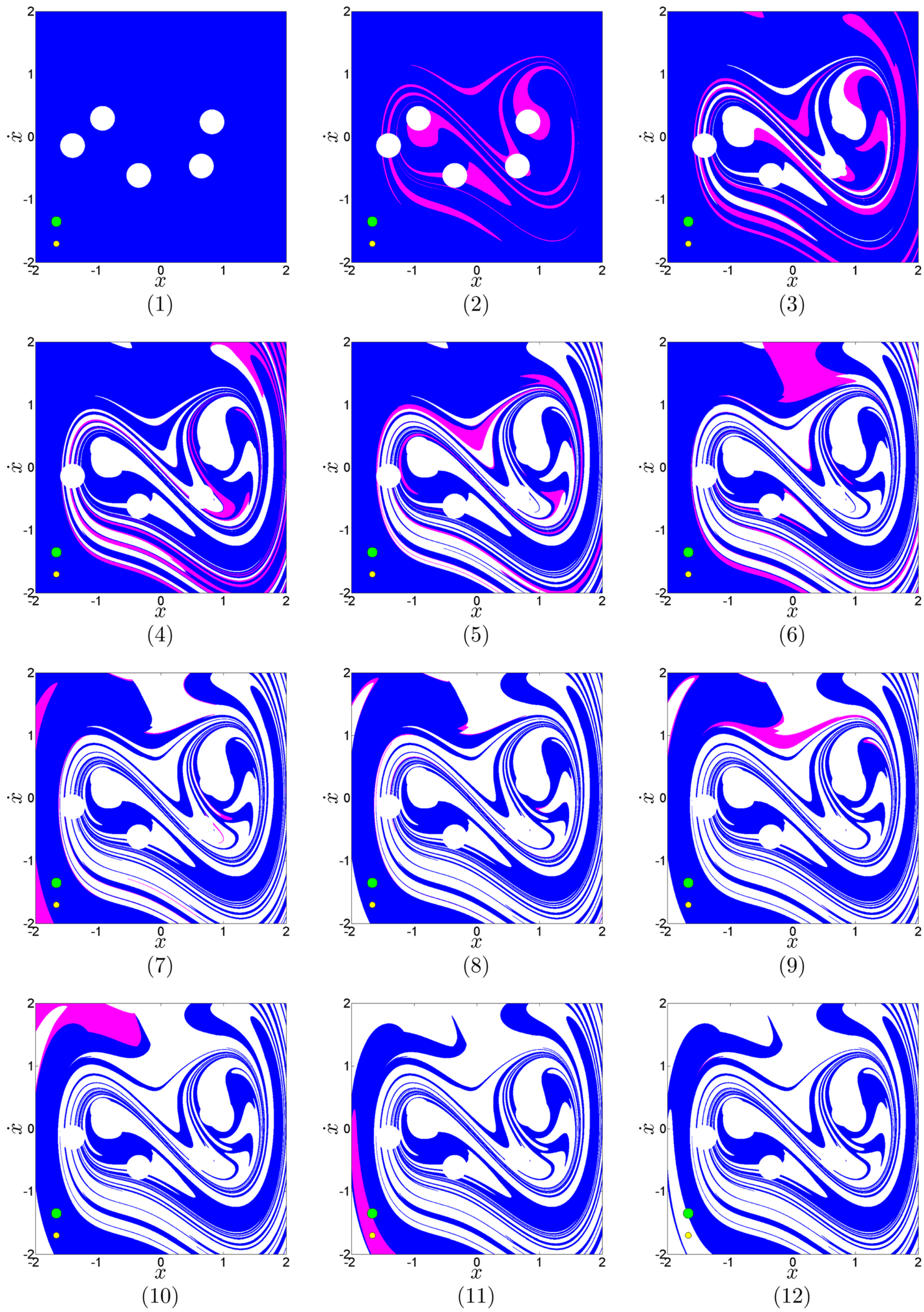

FIG. 5. Safe set sculpting steps. This figure shows the sequence for creating the safe set shown in Fig. 1. At each step, part of Q is removed. The blue color represents the part of the set that remains and the magenta the part that is to be removed.

its points has an S-successor because S is safe, but that point must be in B_∞ since it is asymptotic. The same asymptotic set is shown in red in Fig. 1.

Notice that we start with some closed set B_1 that is a subset of $\rho_S(B_1)$ and a single-point set is quite acceptable. Then $B_n \subset B_{n+1}$. That is, the set B_n grows larger as n increases. Notice also that since the sets B_n are growing in size, the asymptotic set is the closure of their union.

Potentially there could be several asymptotic sets inside one safe set.

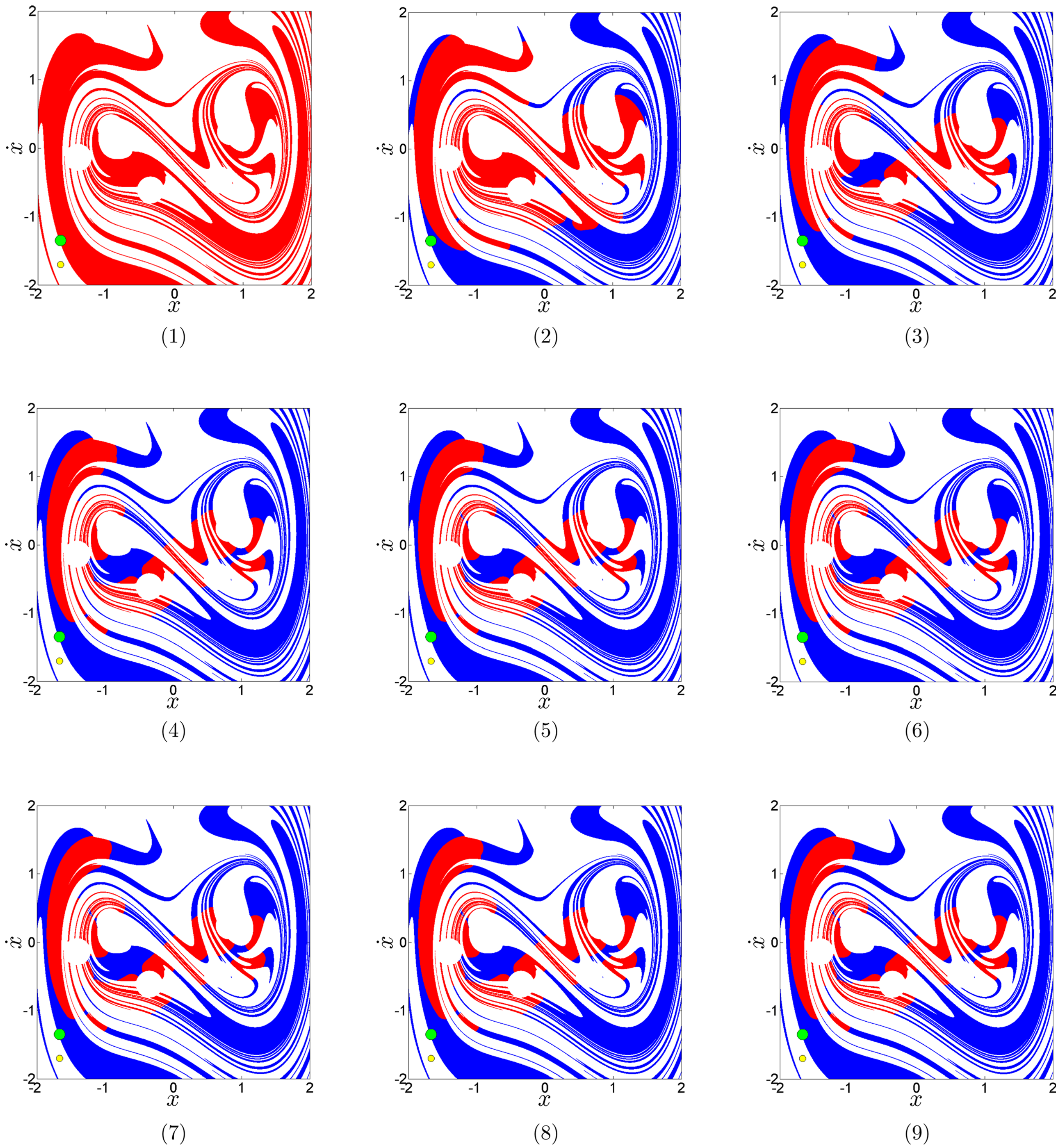

FIG. 6. A safe trajectory can be at time k, shown in panel k. This figure shows the sculpting sequence for creating the asymptotic safe set for the time-2π map of the Duffing oscillator. The safe set is plotted in blue. We start with the whole safe set as initial set, shown in red in panel (1). Then, we start sculpting and removing the parts of the safe set (panel $2, \cdots, 8$) that are not part of the asymptotic safe set. As we iterate the sculpting algorithm, the red set shrinks, converging toward the sculpted asymptotic safe set (shown in panel (9)).

In practice we also compute the asymptotic safe sets using the growing algorithm on a grid. The set function ρ_S used to grow an asymptotic safe set is the same one used to sculpt it. The only difference is the initial set taken. When we sculpt an asymptotic set, we take the initial set to be the entire safe set, while when we grow it we take the initial set to be a small set in the safe set.

If B_1 is not a subset of B_2 but is a subset of B_k, then $B_j \subset B_{j+k-1}$, so we can set $C_n = \cup_{j=n}^{j=n+k} B_j$. Then $C_n \subset C_n + 1$ and $C_{n+1} = \rho_S(C_n)$. Then defining C_∞ to be the closure of the infinite union $\cup_n C_n$ is an invariant set. Notice that $\cup_n B_n = \cup_n C_n$.

Growing a Duffing oscillator asymptotic safe set. Let B_1 be the small red ball that we see in Fig. 7. Then we "grow" B_n as described above, intersecting with Fig. 7(4),

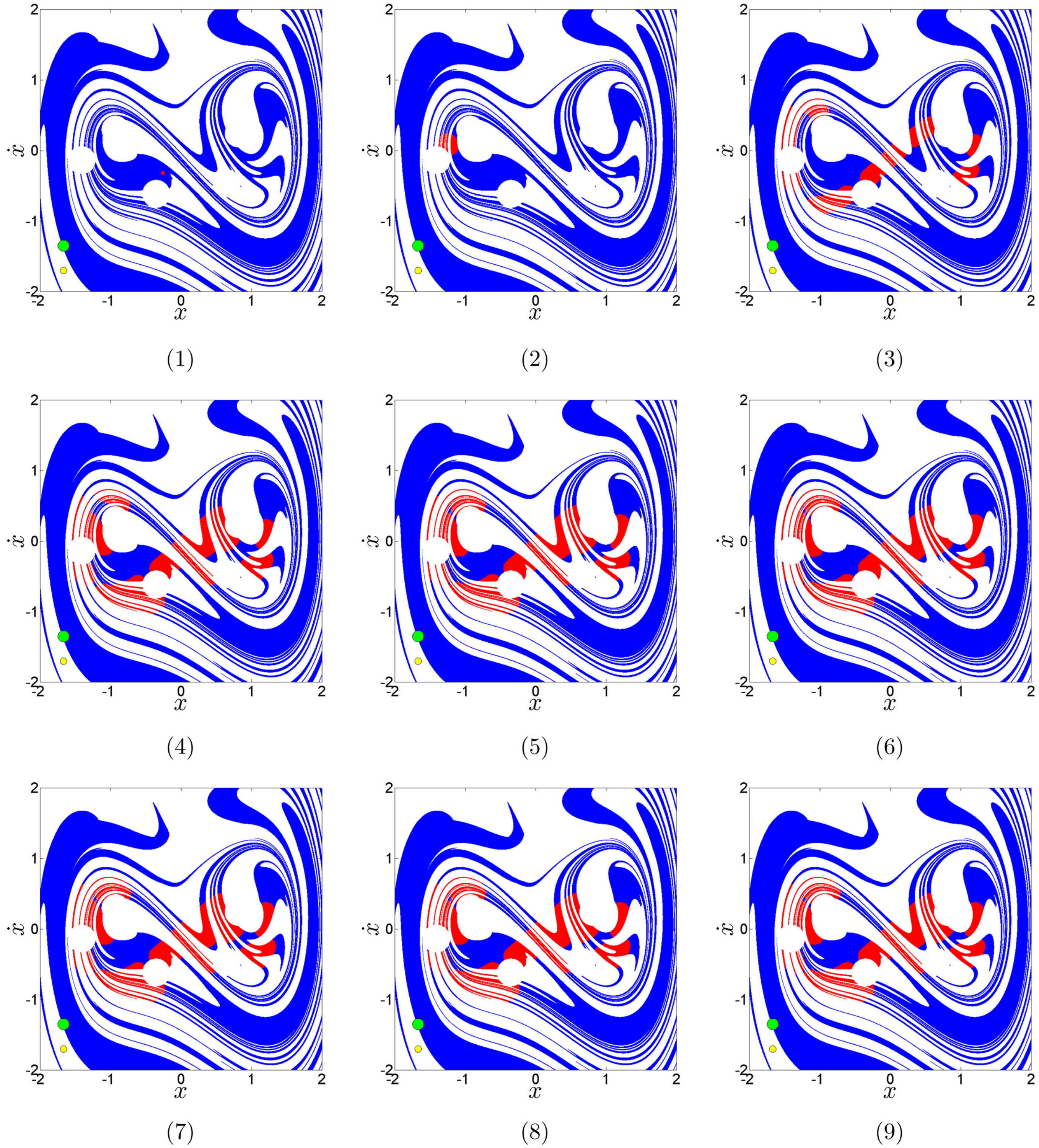

FIG. 7. Growing an asymptotic safe set. This figure shows the sequence growing the asymptotic safe set for the time-2π map of the Duffing oscillator. The safe set is plotted in blue and red. We start with the small red ball shown in panel (1), and we allow all the possible combinations of control and noise provided the point lands in the safe set. As we iterate the growing algorithm the red set expands until finally it converges toward the smallest possible asymptotic safe set, given by panel (9).

we say the safe set S on each step. Comparing Fig. 7(1) with Fig. 7(3), we see that $B_1 \subset B_3$ so $C_1 = B_1 \cup B_2$ will grow with $C_n \subset C_{n+1}$ for all $n > 0$. We can see the final result in Fig. 7(9). The condition "$B_1 \subset B_3$" will always occur if B_1 consists of a single point and that point is a period-two point.

As we can see here, the asymptotic safe set found with the growing procedure is not equal to the asymptotic safe set found with the sculpting procedure, that is, the red set in Fig. 6(9) is bigger (and contains) the red set in Fig. 7(9).

IV. CONCLUSION

We hope that this work will launch efforts to understand how it is possible to use small controls in chaotic (or even stochastic) environments to defeat large occasional disturbances.

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